ASYMPTOTIC OF THE DISPLACEMENT FIELD IN CONTINUOUSLY INHOMOGENEOUS ELASTIC MEDIA

G. P. Kovalenko

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Exact and approximate equations of motion of an elastic continuously inhomogeneous medium are obtained in this paper in the form of coupled wave equations and the displacement asymptotic in a weakly inhomogeneous elastic half-space subjected to tangential and normal loads dependent harmonically on time. The influence of the inhomogeneity of the medium on the displacement asymptotic, the group velocity, and the Rayleigh wavelength, as well as the intensity of the wave interaction energy are expressed in terms of the coefficients connecting the wave equation.

1. Linear wave interaction in inhomogeneous media can be described in many cases [1] by a coupled system of wave equations of the type

$$\left(\nabla^2 - v_n^{-2}\frac{\partial^2}{\partial t^2}\right)\Phi_n - Z_n\Phi_{3-n} = 0,\ n = 1, 2, \tag{1.1}$$

where Z_n are known functions, the desired functions Φ_n can have different meanings, and the other notation is standard. It is interesting to obtain an equation of the type (1.1) for an elastic medium whose Lamé parameters λ, μ and density ρ depend on one coordinate. The vector equation of motion of such a medium has the form

$$\nabla(\varepsilon\nabla\cdot\mathbf{u}) - \nabla\times(\mu\nabla\times\mathbf{u}) + 2\mu'(\mathbf{u}' - \mathbf{i}_z\nabla\cdot\mathbf{u} + \mathbf{i}_z\times\nabla\times\mathbf{u}) = \rho\ddot{\mathbf{u}},\ \varepsilon = \lambda + 2\mu. \tag{1.2}$$

Here $\mathbf{u}$ is the displacement vector; the prime denotes the derivative with respect to the Cartesian coordinate z and the dot with respect to the time; $\mathbf{i}_z$ is the unit vector, and ∇ is the nabla operator in the Cartesian coordinate system.

Applying the Fourier transformation in the variables x and y to (1.2), we obtain a system of ordinary, sixth-order, differential equations with fundamental and perturbing matrices. The fundamental matrix does not contain derivatives of the medium parameters. The efficiency of matrix integration algorithms for such systems [2] depends substantially on the possibility of finding the eigenvalues of the fundamental matrix in explicit and compact form. In this case the eigenvalues are found after solving a cubic equation and are sufficiently awkward; consequently, the matrix integration algorithms are of low efficiency. The method of suspended potentials permits a system of equations without the mentioned disadvantage to be obtained. Since the method mentioned has substantially been touched upon in certain conferences [3, 4] in discussions of similar questions, it is elucidated quite briefly below in an amount necessary for the boundary-value problem under consideration. Let us represent the displacement vector u in the form

$$\mathbf{u} = f_1^{-1}\nabla(f_1\Phi_1) + f_2^{-1}\nabla\times\nabla\times(\mathbf{i}_z f_2\Phi_2) + \nabla\times(\Phi_3\mathbf{i}_z), \tag{1.3}$$

where f_n are still unknown functions of the variable z. Let us substitute (1.3) into (1.2) and append two identities containing functions h_n to the result of the substitution:

$$\nabla(h_1\Phi_1') - \mathbf{i}_z\left(h_1\nabla^2 + h_1'\frac{\partial}{\partial z}\right)\Phi_1 - (\nabla\times\nabla\times\mathbf{i}_z h_1 - h_1'\nabla_t)\Phi_1 \equiv 0,$$

$$\nabla(h_2\nabla_t^2\Phi_2) - \left(\mathbf{i}_z h_2'\nabla_t^2 - \nabla\times\nabla\times h_2\mathbf{i}_z\frac{\partial}{\partial z}\right)\Phi_2 - \nabla_t(h_2\nabla^2\Phi_2 + h_2'\Phi_2') \equiv 0,$$

$$\nabla_t^2 = \nabla_t\cdot\nabla_t,\ \nabla_t = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}.$$

After certain manipulations, we obtain the vector equation

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$$\nabla\Lambda_{11} + \nabla\times\nabla\times\mathbf{i}_z\Lambda_{22} + \nabla\times\mathbf{i}_z\Lambda_{33} - \nabla_t\Lambda_{12} - \mathbf{i}_z\Lambda_{13} = 0 \tag{1.4}$$

(Λ_{nm} are differential expressions dependent on Φ_n) which will be satisfied if the potentials Φ_n are found from the system of scalar equations

$$\Lambda_{11} + \Lambda_{22}' - \Lambda_{12} = 0,\ \Lambda_{11}' - \nabla_t^2\Lambda_{22} - \Lambda_{13} = 0, \\ \Lambda_{33} \equiv \mu\nabla^2\Phi_3 + \mu'\Phi_3' - \rho\ddot{\Phi}_3 = 0. \tag{1.5}$$

We select the values of the functions h_n, $g_n = f_n' f_n^{-1}$ such that the expressions for Λ_{nm} are as simple as possible. If we take $h_1 = h_2 = -2\mu'$, $g_1 = g_2 = \rho'\rho^{-1}$, then

$$\Lambda_{nn} = \chi_n\{\Box\Phi_n + Q_n[(2-n)\nabla_t^2 + 1 - n]\Phi_{3-n}\} \quad (n = 1, 2),$$

$$\Box_n = \nabla^2 + p_\rho\frac{\partial}{\partial z} + p_\rho' - v_n^{-2}\frac{\partial^2}{\partial t^2},\ \chi_1 = \varepsilon,\ \chi_2 = \mu,$$

$$\Lambda_{12} = Z_1\nabla_t^2\Phi_2,\ \Lambda_{13} = Z_1\Phi_1,\ Z_1 = 2\mu' p_\rho - 2\mu'',$$

$$Q_n = p_\rho - 2p_\mu\gamma^{-2(2-n)},\ p_\rho = \rho'\rho^{-1},\ p_\mu = \mu'\mu^{-1},\ \gamma^2 = \varepsilon\mu^{-1}.$$

The third equation in the system (1.5) is separated from the first two, each of which is of third order. The system of ordinary differential equations corresponding to (1.5) is integrated effectively since the characteristic polynomial of its fundamental matrix dissociates into binomial quadratic factors. For a weakly inhomogeneous medium the first approximation equations, i.e., (1.5), can be used, wherein the terms Λ_{12}, Λ_{13} are discarded as being of second order of smallness. We then obtain a system of approximate equations

$$\nabla^2\Phi_1 + (p_\rho\Phi_1)' - v_1^{-2}\ddot{\Phi}_1 + Q_1\nabla_t^2\Phi_2 = 0, \\ \nabla^2\Phi_2 + (p_\rho\Phi_2)' - v_2^{-2}\ddot{\Phi}_2 - Q_2\Phi_1 = 0, \\ \nabla^2\Phi_3 + p_\mu\Phi_3' - v_2^{-2}\ddot{\Phi}_3 = 0. \tag{1.6}$$

2. We use (1.6) below to find the asymptotic of the displacement field in an elastic weakly inhomogeneous half-space if the boundary conditions

$$\sigma_n = \psi_n(x, y)\,e^{i\omega t},\ n = 1, 2, 3,\ x, y \in \Omega, \\ \sigma_1 = \sigma_z,\ \sigma_2 = \sigma_{xz},\ \sigma_3 = \sigma_{yz}, \tag{2.1}$$

are given in the plane $z = 0$, where Ω is a bounded domain of load application, σ_n are stress tensor components, ω is the vibration frequency. The factor $e^{i\omega t}$ is henceforth omitted. The solution of the system (1.6), bounded as $z \to \infty$, is written in Fourier transform space in the variables x, y in a first approximation, i.e., with the components containing the first derivatives with respect to z of the medium parameters conserved. The following operations are here satisfied. After the Fourier transformation, the first two equations of (1.6) (the third is integrated independently) are written in the form of a system of four first-order equations whose eigenvalues of the fundamental matrix equal $\pm(\alpha^2 + \beta^2 - k_n^2)^{1/2}$, $k_n = \omega v_n^{-1}$, where α, β are Fourier transform parameters. The system is later subjected to matrix transformation, after which the fundamental matrix becomes quasi-diagonal since it is assumed that its eigenvalues have one point of rotation. Then the second transforming matrix, which assures splitting of the system into two independent pairs of equations [5], is constructed in a first approximation. Finally, the method of standard systems is applied to each pair and to the Fourier transform of the third equation from (1.6), whereupon

$$\mathbf{\Phi} = W\mathbf{C},\ \mathbf{\Phi} = \{\Phi_1, \Phi_1', \Phi_2, \Phi_2', \Phi_3, \Phi_3'\}^t, \tag{2.2}$$

$$W = \left\|\begin{matrix} \left\|\begin{matrix} W_1 & W_1' & d_2W_2 & d_2W_2' \\ d_1W_2 & d_1W_2' & W_2 & W_2' \\ 0 & 0 & 0 & 0 \end{matrix}\right\| \sqrt{\frac{\rho_0}{\rho}} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ \sqrt{\frac{\mu_0}{\mu}}\,\|W_2 W_2'\| \end{matrix} \end{matrix}\right\|^t,$$

$$W_n = \frac{w\left(k^{3/2}\varphi_n\right)}{\sqrt{\varphi_n'}},\ \varphi_n = \left(\frac{3}{2}\int_{z_n}^{z} m_n(z)\,dz\right)^{2/3},\ \rho_0 = \rho(0),$$

$$m_n = (\zeta^2 + \xi^2 - v_0^2 v_n^{-2}(z))^{1/2},\ d_n = (-1)^n\frac{Q_n}{1-\gamma^{-2}}v_n^2(z)\,v_2^{-2}(0),$$

$$C = (c_1, c_2, c_3)^t, \zeta = \alpha k^{-1}, \xi = \beta k^{-1}, k = \omega v_2^{-1}(0).$$

Here w is the Airy function bounded as $z \to \infty$, W_n is the Airy–Fok function, and the superscript t indicates the transposition operation. The constants c_n are found from the Fourier transformed boundary conditions (2.1)

$$\mu PWC = \sigma, \ \sigma = (\sigma_1, \sigma_2, \sigma_3)^t,$$

where

$$P = \begin{Vmatrix} 2i\alpha q & 2i\alpha & i\alpha\varkappa^2 & 0 & 0 & i\beta \\ 2i\beta q & 2i\beta & i\beta\varkappa^2 & 0 & 0 & i\alpha \\ \varkappa^2 & 0 & 2\eta^2 q & 2\eta^2 & 0 & 0 \end{Vmatrix};$$

$$\eta^2 = \alpha^2 + \beta^2; \ \varkappa^2 = 2\eta^2 - k_2^2; \ q = p_\rho + Q_2.$$

The Fourier transforms in (2.2) are denoted exactly as are the originals. We determine the arbitrary constants for two cases. In the boundary conditions (2.1) let the normal stress differ from zero. Then

$$\mathbf{u} = (2\pi)^{-2} \int\limits_{-\infty}^{\infty}\int SWCe^{i(\alpha x + \beta y)} d\alpha\, d\beta, \tag{2.3}$$

$$S = \begin{Vmatrix} i\alpha & 0 & i\alpha p_\rho & i\alpha & i\beta & 0 \\ i\beta & 0 & i\beta p_\rho & i\beta & -i\alpha & 0 \\ p_\rho & 1 & \eta^2 & 0 & 0 & 0 \end{Vmatrix},$$

$$c_1 = \psi_1(\alpha, \beta) L (\mu_0 \Delta)^{-1}, \ c_2 = -\psi(\alpha, \beta) K (\mu_0 \Delta)^{-1},$$

$$K = 2W_1' + W_1(2q + \varkappa^2 d_2), \ L = \varkappa^2 W_2 + 2d_1\eta^2 W_2,$$

$$\Delta = \varkappa^4 W_1 W_2 - 4\eta^2 W_1' W_2' + 2\tau^2\varkappa^2 (W_2 W_1' - W_1 W_2')(d_2 - d_1) -$$
$$- 4q\eta^2 (W_1 W_2' + W_2 W_1'), \ \tau^2 = \zeta^2 + \xi^2, \ \mu_0 = \mu(0).$$

If only the tangential stresses differ from zero, then we find the displacement by means of the same dependences (2.3) but for other values of the constants c_n which are now determined as

$$c_n = \pi_1 E_n (\mu_0 \Delta)^{-1} \ (n = 1, 2), \quad c_3 = \pi_2 (W_2')^{-1},$$

$$\pi_1 = (\psi_2(\alpha, \beta)\alpha + \psi_3(\alpha, \beta)\beta)(\alpha^2 + \beta^2)^{-1}, \ E_1 = 2\eta^2 W_2' + (2q\eta^2 + \varkappa^2 d_1) W_2,$$

$$E_2 = \varkappa^2 W_1 + 2\eta^2 d_2 W_1',$$

$$\pi_2 = -i(\psi_2(\alpha, \beta)\beta - \psi_3(\alpha, \beta)\alpha)\eta^{-2}.$$

In the simplest case of constant stresses σ_n within a domain in the shape of a rectangle $|x| \le a$, $|y| \le b$, the factors are $\psi_n = \sigma_n \sin(\alpha a)\sin(\beta b)(\alpha\beta)^{-1}$. We analyze the displacement field (2.3) in the same domain of variables where the Airy–Fok functions allow the asymptotic representation

$$W_n = m_n^{-1/2} \exp\left(-k \int\limits_{z_n}^{z} m_n dz\right), \quad n = 1, 2$$

(z_n is the point of rotation). After the substitution mentioned, we calculate the principal term of the asymptotic of the integral obtained in a remote point of the field. To do this it is necessary to evaluate the contribution of three kinds of points: saddle, bifurcation, and the poles of the integrand. But since the contribution of the bifurcation points is an order below the contribution of the saddle points [6], it is not considered here. We approximate saddle points by considering the medium to be slightly inhomogeneous. In the phase function

$$-k \int\limits_0^z m_n(z)\, dz + i(\alpha x + \beta y)$$

we calculate the integral by the method of freezing and we go over to variables R, θ, φ of a spherical coordinate system. Then we obtain for the saddle points

$$\zeta_n = -q_n \sin\theta \cos\varphi, \quad \xi_n = -q_n \sin\theta \sin\varphi, \quad q_n = v_2(0) v_n^{-1}(z).$$

We note the contribution of the saddle points to the displacement field by the subscript s. After the calculations specified by the two-dimensional stationary phase method, we express the displacement of an elastic medium in a spherical coordinate system by means of the formulas

$$\begin{pmatrix} u_R \\ u_\theta \end{pmatrix} = (u_x \cos\varphi + u_y \sin\varphi)\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \pm u_z \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$
$$u_\varphi = -u_x \sin\varphi + u_y \cos\varphi.$$

We then obtain for the displacements

$$u_{ns} = \sum_{\nu=1}^{2} G_\nu K_{n\nu}, \quad u_{1s} = u_{Rs}, \quad u_{2s} = u_{\theta s}, \quad u_\varphi = 0, \tag{2.4}$$
$$G_\nu = \frac{q_\nu \cos\theta}{2\pi Rk} \exp\left(-ikR\gamma^{\nu-2} + \frac{\pi i}{2}\right)\left(\frac{m_\nu(0)}{m_\nu(z)}\right)^{1/2} \frac{\psi_1(\zeta_\nu, \xi_\nu)}{F(\zeta_\nu, \xi_\nu)},$$
$$K_{11} = k^3(\mu_1\varkappa_{1\theta} - 2\delta_1), \quad \mu_n = kq_n i - p_\rho \cos\theta,$$
$$K_{22} = k^3 q_2 \sin^2\theta (2\vartheta_2\mu_2 - q_2 T_2), \quad T_n = 2q(0) + \varkappa_{n\theta} d_n,$$
$$\begin{pmatrix} K_{12} \\ K_{21} \end{pmatrix} = \frac{k^3\gamma^2 \sin\theta}{\gamma^2 - 1}\begin{pmatrix} -2Q_2\vartheta_2 q_2 \\ \varkappa_{10}^2 Q_1 \end{pmatrix}, \quad \varkappa_{n\theta}^2 = 2q_n^2 \sin^2\theta - 1,$$
$$\delta_n = 2q_n\vartheta_n d_n(0)\sin^2\theta, \quad \vartheta_n = \left(\gamma^{-2(n-1)} - q_n^2 \sin^2\theta\right)^{1/2},$$
$$F(\zeta, \xi) = k^3[(\varkappa^4 - 4\tau^2 m_1 m_2)k + 2(d_2 - d_1)\varkappa^2\tau^2(m_2 - m_1) + 4\tau^2 q(m_1 + m_2)],$$

where $\psi_1(\zeta_\nu, \xi_\nu)$ is the Fourier transform of the function ψ_1 evaluated at the saddle point. Under the action of a tangential load we should take in place of ψ_1, K_{nk} the π_1, L_{nk} and the displacement u_φ will be nonzero:

$$\begin{pmatrix} L_{12} \\ L_{21} \end{pmatrix} = \frac{k^3 \sin^2\theta\gamma^2}{\gamma^2 - 1}\begin{pmatrix} q_2\varkappa_{2\theta}Q_2 \\ \vartheta_1 q_1^2 Q_1 \end{pmatrix},$$
$$L_{11} = k^3 q_1 \sin^2\theta (2\mu_1\vartheta_1 + q_1 T_1),$$
$$L_{22} = ik^3 q_2^2 \sin\theta (\mu_2\varkappa_{2\theta} + 2\delta_2),$$
$$u_\varphi = i\sin\theta q_2^2 \pi_2(\zeta_2, \xi_2)\frac{\cos\theta}{2\pi R} e^{-\left(kR + \frac{\pi}{2}\right)} (m_2(0)\, m_2(z))^{-1/4}.$$

Let us make a number of deductions. In a homogeneous medium the displacements u_R and u_θ are propagated far from the source as longitudinal and transverse waves, respectively. In this case the longitudinal wave in the displacement u_R is accompanied by a transverse wave that vanishes in the case of a homogeneous medium. Consequently, the fundamental wave is longitudinal while the transverse can be called induced. The fundamental wave is transverse in the displacement u_θ and the longitudinal is induced. The amplitudes of the induced waves are proportional to the coupling coefficients of the wave equations Q_n. As is known [6], the principal term of the displacement asymptotic u_θ is real for a homogeneous medium while it can be both real and complex in the displacement u_θ. Inhomogeneity of the medium makes both the expressions mentioned complex; in particular, the Rayleigh function is complex at both saddle points. For instance, we have at the saddle point ζ_1, ξ_1

$$F = k^3(F_{11}k + iF_{12}), \quad F_{11} = \varkappa_{1\theta}^4 + 4q_1^2\vartheta_1\vartheta_2 \sin^2\theta,$$
$$F_{12} = 2q_1^2 \sin^2\theta\left[\varkappa_{1\theta}^2 (d_2 - d_1)(\vartheta_2 - \vartheta_1) + 2q(\vartheta_2 + \vartheta_1)\right].$$

This means that the maximal radial and circumferential displacements are achieved at different points of the wave surface at different times; the difference in the time depends not only on the mutual arrangement of the two points but also on the nature of the wave propagation. Thus, for waveguide propagation, when v_1 grows, ϑ_1 remains a real quantity. If v_1 decreases, then ϑ_1 becomes imaginary for $q_1 \sin\theta > \gamma^{-1}$ and the form of the function F changes.

Inhomogeneity of the medium results in energy redistribution of the vibrations between the longitudinal, transverse, and surface waves. Part of the energy goes into the formation of induced waves. Let us derive the formula for the interaction energy intensity for the fundamental and induced waves. As is known the energy emission intensity is defined as the mean emission energy per period per unit area and is found from the formula

$$-\frac{i\omega}{4}(\sigma_R\bar{u}_R - \bar{\sigma}_R u_R + \sigma_{R\theta}u_\theta - \bar{\sigma}_{R\theta}u_\theta),$$

where the bar denotes the complex-conjugate quantity and it is assumed that $u_\varphi = 0$. If the displacement is written in the form

$$u_R = (A_1 + iB_1)F^{-1}(\zeta_1, \xi_1) + V_1Q_2F^{-1}(\zeta_2, \xi_2) \equiv u_{Rl} + u_{Rs},$$
$$u_\theta = (A_2 + iB_2)F^{-1}(\zeta_2, \xi_2) + V_2Q_1F^{-1}(\zeta_1, \xi_1) \equiv u_{\theta s} + u_{\theta l} \quad (2.5)$$

(the second components on the right correspond to the induced waves), then it can be shown that in a remote point of the field

$$\sigma_R = \mu\gamma^2 i(\gamma^{-1}u_{Rl} + u_{Rs})k + O(R^{-2}),$$
$$\sigma_\theta = \mu i(\gamma^{-1}u_{\theta l} + u_{\theta s})k + O(R^{-2}).$$

Here $O(R^{-2})$ indicates the order of magnitude of the discarded components. Then we have for the energy intensity in a first approximation

$$k\frac{\omega\mu}{4}\{2\gamma u_{Rl}\bar{u}_{Rl} + 2u_{\theta s}\bar{u}_{\theta s} + (1+\gamma^{-1})[(u_{Rl}\bar{u}_{Rs} + \bar{u}_{Rl}u_{Rs})\gamma^2 + u_{\theta l}\bar{u}_{\theta s} + \bar{u}_{\theta l}u_{\theta s}]\}.$$

The second component in the formula obtained yields the energy intensity of interaction between the fundamental and induced waves. Let us write it in greater detail, for which we introduce two plane vectors $N = F_{n1} + iF_{n2}$ whose components equal the real and imaginary parts of the Rayleigh function evaluated at the saddle points. Then the component mentioned takes the form

$$k\frac{\omega}{4}\mu(1+\gamma^{-1})\sum_{n=1}^{2}\frac{Q_nV_n\gamma^{2(2-n)}(B_nN_1\cdot N_2 + (-1)^nA_n|N_1\times N_2|)}{|N_1|^2|N_2|^2},$$

where A_n, B_n, V_n are found by comparing (2.4) and (2.5).

To estimate the displacements caused by the Rayleigh wave in a remote point of the field we find the root of the function $F(\zeta, \xi)$ corresponding to the surface wave. If the value b_0 of the root for a homogeneous medium is taken as the zeroth approximation, then we obtain by the Newton method for the first approximation

$$b_1 = b_0(1-M), \quad M = M_1M_2^{-1},$$
$$M_1 = 2b_0^{-2}(2-b_0^2)(d_2-d_1)(\vartheta_{2b}-\vartheta_{1b}) + 4q(\vartheta_{2b}+\vartheta_{1b}),$$
$$M_2 = \Pi_1k + \Pi_2, \quad \Pi_1 = 4b_0(\vartheta_{2b}\vartheta_{1b}^{-1}\gamma^{-2} + \vartheta_{1b}\vartheta_{2b}^{-1} - 2 + b_0^2),$$
$$\Pi_2 = M_1 - 2(d_2-d_1)[2(\vartheta_{2b}-\vartheta_{1b}) + (2-b_0^2)(\gamma^{-2}\vartheta_{1b}^{-1} - \vartheta_{2b}^{-1})] -$$
$$-4qb_0^2(\vartheta_{1b}^{-1}\gamma^{-2} + \vartheta_{2b}^{-1}), \quad \vartheta_{nb} = (1 - b_0^2\gamma^{2(n-2)})^{1/2}.$$

We hence find the velocity increment Δv_R and the length increment Δl_R of the Rayleigh wave caused by inhomogeneity of the medium

$$\Delta v_R = v_R - v_{R0} = b_0Mv_2(0), \quad \Delta l_R = 2\pi\omega^{-1}b_0Mv_2(0)$$

(the subscript 0 indicates the value of the quantity for a homogeneous medium). We find the group velocity v_{Rg} of the Rayleigh wave by differentiating the dependence $\omega = k_Rv_R$ with respect to k_R, the Rayleigh wave number:

$$v_{Rg} = \frac{d\omega}{dk_R} = v_R(1-M)(1 - Mk_R^{-1}M_2^{-1})^{-1}.$$

As should have been expected, as the frequency of vibrations grows, the influence of the inhomogeneity decreases and the group velocity tends to the phase velocity in the limit. To estimate the displacement field caused by the Rayleigh wave at a remote point of the field, we evaluate the inner integral in (2.3) by the theorem on residues, and the outer by the stationary phase method. Here $\zeta = \zeta_R = (c_R^2 - \xi^2)^{1/2}$, $c_R = v_{20}v_R^{-1}$ in the Rayleigh root, and the stationary value of ξ is found at the point $\xi_R = c_R \sin\varphi$, $\varphi = \tan^{-1}(yx^{-1})$. As an illustration we present the asymptotic of the axial displacement u_{zR} at the point $z = 0$ under the action of a normal stress

$$u_{zR} = G_R[m_{1R}k + 2p_\mu c_R^2 - p_\rho], \quad r = (x^2+y^2)^{1/2},$$
$$G_R = 2\pi\psi_1(\zeta_R, \xi_R)\left(\frac{2\pi c_R}{Rk}\right)^{1/2}\exp(i\omega rv_R^{-1} - i\omega t)\Gamma^{-1}(c_R),$$
$$\Gamma(c_R) = \zeta^{-1}\frac{\partial F}{\partial\zeta}, \quad \zeta = \zeta_R, \quad m_{1R} = (c_R^2 - v_2^2(0)v_1^{-2}(0))^{1/2}.$$

In the case of a constant stress σ_1 applied to the boundary of a rectangular area

$$\psi_1(\zeta_R, \xi_R) = k_R^{-2} \sin^{-1} \varphi \cos^{-1} \varphi \sin(k_R a \cos\varphi) \sin(k_R b \sin\varphi) \sigma_1.$$

Hence it follows that if the size of the area is selected according to the dependences $a = n\pi v_R/\omega \cos\varphi$ or $b = n\pi v_R/\omega \sin\varphi$, then there will be no displacements. Therefore, the influence of the inhomogeneity of the medium on the size of the area for which the maximal or minimal part of the energy of the vibrations source is transmitted to excitation of a Rayleigh wave in a given direction can be estimated.

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ANTIPLANAR DEFORMATION OF AN ELASTOPLASTIC STRIP WITH A SEMI-INFINITE CRACK

V. G. Novikov

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The features and details of plastic flow at a crack tip govern its development [1]. Therefore, it is important to have a correct idea about the shape and dimensions of the plastic zone, and about the intensity of deformation in it. In view of this there is considerable importance in the problem during whose solution, apart from determining stresses and strains, there should be determination without prior assumptions of the boundary separating the elastic and plastic regions. A study was made in [2-8] of approximate and numerical methods for this problem, and analytical solutions in closed form have only been obtained for antiplanar deformation of a boundless material with one rectilinear crack or a periodic system of collinear defects [9-14]. In this work an accurate solution is obtained for the elastoplastic problem of antiplanar deformation of a strip with a semi-infinite crack.

We consider antiplanar deformation of a strip made of elastoplastic material occupying the region $|x| < \infty$, $|y| \leq d$. It is assumed that in the plastic condition material behavior is described by the Tresk condition

$$\sigma_{xz}^2 + \sigma_{yz}^2 = \tau_*^2 \tag{1}$$

and by an associated rule for plastic flow, and in the elastic region by Hooke's linear rule

$$\sigma_{xz} = \mu \frac{\partial w}{\partial x}, \quad \sigma_{yz} = \mu \frac{\partial w}{\partial y}, \tag{2}$$

$$\sigma_{xz}^2 + \sigma_{yz}^2 < \tau_*^2, \tag{3}$$

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